RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2603

Pure Mathematics 3

INSTRUCTIONS

Thursday **16 JUNE 2005** Afternoon 1 hour 20 minutes
+ up to 1 hour

The paper is in two parts:

Section A (1 hour 20 minutes)
Section B (up to 1 hour)

Supervisors are requested to ensure that Section B **is not issued** until Section A has been collected in from the candidates.

Centres may, if they wish, grant a supervised break between the two parts of this examination.

Invigilators are not required to match up candidates' two parts. Part A and Part B should be sent to the examiner as two sets of scripts with candidates in the same order as the attendance register for each set.

This notice must be on the Invigilator's desk at all times during the afternoon of Thursday 16 June 2005.

These instructions consist of 1 printed page and 1 blank page.

SP (NF) S83692

RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2603(A)

Pure Mathematics 3

Section A

Thursday **16 JUNE 2005** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this section is 60.

NOTE

- This paper will be followed by **Section B: Comprehension**.

This question paper consists of 4 printed pages.

1 **(a)** Express $5\cos x - 6\sin x$ in the form $R\cos(x + \alpha)$, where $0 < \alpha < \frac{1}{2}\pi$. [4]

(b) Evaluate $\int_0^{\frac{1}{4}\pi} x\sin 2x \, dx$. [5]

(c) A curve is defined implicitly by the equation

$$y + \ln y = \sin x.$$

Verify that the point $(\frac{1}{2}\pi, 1)$ lies on the curve.

Show that $\frac{dy}{dx} = \frac{y\cos x}{y+1}$.

Hence find the gradient of the curve at the point $(\frac{1}{2}\pi, 1)$. [5]

[Total 14]

2 **(i)** Express $\frac{5}{(2+x)(1-2x)}$ in partial fractions. [3]

(ii) The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{5y}{(2+x)(1-2x)},$$

with $y = 2$ when $x = 0$.

Show by integration that

$$y = \frac{2+x}{1-2x}.$$ [6]

(iii) Using a binomial expansion, show that the cubic approximation to y is

$$f(x) = 2 + 5x + 10x^2 + 20x^3.$$

State the range of values of x for which the binomial expansion is valid. [4]

(iv) The value $f'(0.01)$ can be used to approximate $\frac{dy}{dx}$ when $x = 0.01$. Show that this approximation is accurate to 3 decimal places. [3]

[Total 16]

3 Fig. 3 shows the ellipse with parametric equations

$$x = 1 - \cos 2\theta, \quad y = 2\sin 2\theta, \quad \text{for } 0 \leqslant \theta < \pi.$$

The curve crosses the x-axis at O and A. The maximum point is B.

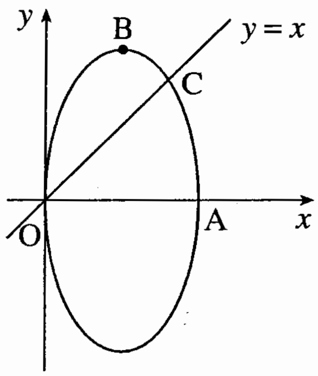

Fig. 3

(i) Find the coordinates of A and B. [4]

(ii) The line $y = x$ meets the curve at the points O and C. Using the double angle formulae, show that, at the point C, $\tan\theta = 2$. [4]

(iii) Show that $\dfrac{dy}{dx} = 2\cot 2\theta$. Hence find the gradient of the curve at the point C. [5]

(iv) Find the cartesian equation for the curve. [2]

[Total 15]

[Turn over

4 Fig. 4 shows a triangular prism. The position vectors of A, B and C, relative to an origin O, are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

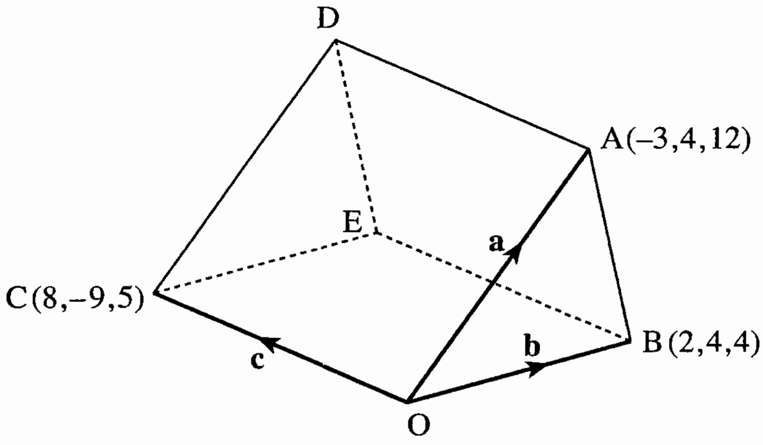

Fig. 4

(i) Write down the vectors $\mathbf{a}$ and $\mathbf{b}$.

Calculate the angle AOB, and find the area of triangle AOB. [6]

The triangle CDE is the triangle OAB translated by the vector $\mathbf{c}$.

(ii) Write down a vector equation of the line AD. [2]

(iii) Show that the vector $\mathbf{c}$ is perpendicular to the plane OAB. Hence find the cartesian equations of the planes OAB and CDE. [5]

(iv) Find the volume of the prism. [2]

[Total 15]

Candidate Name	Centre Number	Candidate Number

RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2603(B)**

Pure Mathematics 3

Section B: Comprehension

Thursday **16 JUNE 2005** Afternoon Up to 1 hour

Additional materials:
Rough paper
MEI Examination Formulae and Tables (MF12)

TIME Up to 1 hour

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces at the top of this page.
- Answer **all** questions.
- Write your answers in the spaces provided on the question paper.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- The insert contains the text for use with the questions.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with your question paper.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this section is 15.

For Examiner's Use	
Qu.	Mark
1	
2	
3	
4	
5	
6	
Total	

This question paper consists of 3 printed pages, 1 blank page and an insert.

For Examiner's Use

1 Explain why the number 1836.108 for the ratio $\frac{\text{Rest mass of proton}}{\text{Rest mass of electron}}$ would be suitable for communication with other civilisations whereas neither the rest mass of the proton nor that of the electron would be. [2]

....

....

....

2 A civilisation which works in base 5 sends out the first 6 digits of π as 3.032 32. Convert this to base 10. [2]

....

....

....

....

3 Complete this table to show the next 3 values of the iteration

$$x_{n+1} = kx_n(1 - x_n)$$

in the case when $k = 3.2$ and $x_0 = 0.5$. Give your answers to calculator accuracy. [1]

n	x_n
0	0.5
1	0.8
2	0.512
3	
4	
5	

For Examiner's Use

4 Justify the statement that the equation in line 83,

$$\frac{\phi}{1} = \frac{1}{\phi - 1},$$

has the solution $\phi = \dfrac{1 \pm \sqrt{5}}{2}$. [2]

....

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5 A sequence is defined by

$$a_{n+1} = 2a_n + 3a_{n-1} \quad \text{with } a_1 = 1 \text{ and } a_2 = 1.$$

Using the method on page 5, show that the value to which the ratio of successive terms converges is 3.

[4]

....

....

....

....

....

....

6 Use the information in the article, including the value of Feigenbaum's number given in line 142, to predict an approximate value of k at which the bifurcation from 8 to 16 outcomes occurs for the iterative equation

$$x_{n+1} = kx_n(1 - x_n).$$ [4]

....

....

....

....

....

....

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2603(B)

Pure Mathematics 3

Section B: Comprehension

INSERT

Thursday **16 JUNE 2005** Afternoon Up to 1 hour

INSTRUCTIONS TO CANDIDATES

- This insert contains the text for use with the questions.

This insert consists of 8 printed pages.

HN/2

Communicating with other civilisations

Background

From time immemorial, people have looked into the night sky and wondered whether there are other civilisations out there. During the last hundred years, it has become a theoretical possibility that, if such civilisations do exist, we could communicate with them.

This article looks at some ideas as to how such communication might get started.

First contact

Imagine then that we have reason to think that a planet orbiting a star some light years away might be home to intelligent life. We have no idea what form that life might take, let alone how their society might work.

Natural curiosity means that we would try to make contact with them by sending some sort of message. There would be two requirements.

(i) The message would have to be such that it could be understood anywhere, and so free of any human or Earth-related influence.

(ii) We would want to know whether the message had been received and understood, so it would have to invite a reply.

Many people believe that mathematics is the only universal culture-free language on Earth. So it would be appropriate to use it as a basis for first contact.

One suggestion is that we should transmit the first 5 (say) digits of π, 3.1415. This could be done using short pulses, with a longer one for the decimal point, as illustrated in Fig. 1.

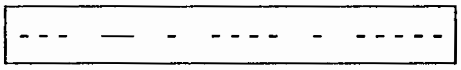

Fig. 1

We would then await a reply consisting of the next 5 digits, 92653.

Will they understand π?

The number we call π arises because it is the ratio of the circumference to the diameter of a circle. It also occurs in many other aspects of mathematics.

An important point is that π requires no units. It is one length divided by another so that, provided the lengths are measured in the same units, the units cancel out. You get the same answer whether you measure the circumference and diameter in metres, feet, miles or anything else. Thus π is a pure number; it is dimensionless.

However, the digits we associate with π are dependent on our number system which uses base 10. If we used base 8, the first 6 digits of π would be 3.11037 since

$$\pi \approx 3 + \frac{1}{8} + \frac{1}{8^2} + \frac{0}{8^3} + \frac{3}{8^4} + \frac{7}{8^5},$$

compared with the base 10 equivalent of

$$\pi \approx 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5}.$$

The reason we use base 10 is probably because we have a total of 10 fingers and thumbs. So another civilisation might well use a different number base. However, it is perhaps reasonable to assume that, if they are intelligent enough to communicate with us, they also have the sense to try out different number bases.

By communicating in base 10, we are telling the other civilisation that the number 10 is for some reason important in our culture.

A reply comes back

We would not, of course, send π just once. The other civilisation would almost certainly miss it! So we would keep on sending it until either we received a reply or we decided to give up. The nearest star is 4.3 light years away, so the soonest we could possibly get a reply would be 8.6 years. If, however, the star of interest was 50 light years away (not far in astronomical terms), the conversation would be even slower, once every 100 years.

The reply would almost certainly consist of two parts: the answer to our question (92653) and a question of their own. This might perhaps be the signal given in Fig. 2, representing the first 5 digits of e, the base of natural logarithms.

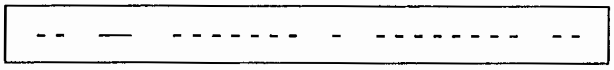

Fig. 2

Receiving this message would be one of the most exciting events in human history. Undoubtedly it would set off a lively debate about what to send next. It is reasonable to conjecture that three groups would be particularly interested.

- Military personnel would want to assess what the outcome would be if we were to find ourselves at war with the other civilisation.
- Others would want to find out about their culture.
- Scientists would hope to learn from them.

Military questions

The military would find themselves in considerable difficulty. They would want to assess the weapons capability of the other civilisation, and so would need to devise a sequence of different numbers (or questions) each of which related to a different stage of weapons development.

However, sending out such a sequence of numbers would be fraught with danger. The other civilisation might well work out the precise purpose of the questions and so deduce our own level of military sophistication without giving away any information from their side. It could well be seen as sending out a hostile message with the possibility that the other civilisation would then break off communication.

So it is quite possible that direct military involvement would be seen as just too risky.

Cultural questions

The golden ratio

A number that would certainly be considered is the *golden ratio*. The sides of the rectangle in Fig. 3 are in the ratio 1.618... to 1. This is called the golden ratio and the number 1.618... is denoted by ϕ (the Greek letter phi). Such a rectangle is called a *golden rectangle*.

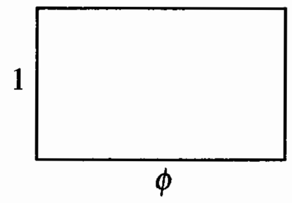

Fig. 3

The golden ratio has considerable cultural significance. For thousands of years a golden rectangle, like the one in Fig. 3, has been regarded as more pleasing to the eye than any other rectangle. So ϕ is related to our artistic sense.

To derive the number ϕ, look at the rectangle ABCD in Fig. 4. It is divided into two parts: a square EBCF at one end, and a smaller rectangle AEFD.

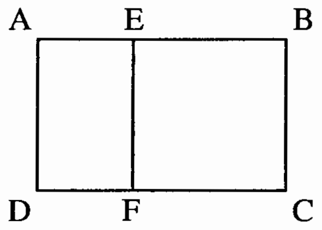

Fig. 4

If the ratio of the sides of the small rectangle AEFD is the same as that of ABCD, then ABCD is a golden rectangle. In this case,

$$\frac{AB}{AD} = \frac{AD}{AE} = \phi.$$

Thus, if the smaller side, AD, of the main rectangle is given a value of 1 unit, the longer side, AB, is ϕ units.

Since EBCF is a square, the length AE is $(\phi - 1)$ units, and so

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

$$\Rightarrow \quad \phi = \frac{1 \pm \sqrt{5}}{2}.$$

Since ϕ must be positive, it follows that the golden ratio is $\dfrac{1+\sqrt{5}}{2}$ or 1.61803… .

Notice that if the longer side of a golden rectangle is assigned a length of 1 unit, the length of the shorter side is $\dfrac{1}{\phi} = \dfrac{\sqrt{5}-1}{2}$ units.

The number $\dfrac{1+\sqrt{5}}{2}$ crops up in many other places in mathematics, including the Fibonacci sequence. This is usually written

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \ldots .$$

The first two terms are both given the value 1. After that, each term is the sum of the previous two terms and so the sequence can be defined iteratively by

$$a_{n+1} = a_n + a_{n-1} \quad \text{with } a_1 = 1 \text{ and } a_2 = 1.$$

The ratios of one term to the previous term form the sequence

$$\frac{1}{1} = 1,\ \frac{2}{1} = 2,\ \frac{3}{2} = 1.5,\ \frac{5}{3} = 1.666\ldots,\ \frac{8}{5} = 1.6,$$
$$\frac{13}{8} = 1.625,\ \frac{21}{13} = 1.615\ldots,\ \frac{34}{21} = 1.619\ldots,\ \ldots .$$

This sequence converges and it looks as though its limit has the same value as ϕ.

To prove that it does, imagine that you have taken so many terms that the ratio has settled down to a value, r.

Thus $$r = \frac{a_{n+1}}{a_n} \quad \text{and} \quad r = \frac{a_n}{a_{n-1}}.$$

Take the equation $a_{n+1} = a_n + a_{n-1}$ and divide through by a_n.

$$\frac{a_{n+1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$$

and so $$r = 1 + \frac{1}{r}$$
$$\Rightarrow \quad r^2 - r - 1 = 0$$
$$r = \frac{1+\sqrt{5}}{2} \quad \left(\text{or } \frac{1-\sqrt{5}}{2}\right).$$

The Fibonacci sequence occurs in nature, for example in connection with the numbers of petals on several types of flowers.

So the message conveyed to another civilisation by the number ϕ would not be unique. They might think we were telling them about our artistic sense, or about the flowers that grow on our planet or about something else.

Feigenbaum's number

A particularly interesting number to send is Feigenbaum's number. This was discovered in 1975 as a result of work on the (then) new subject of chaos.

A simple model for population growth is given by the *logistic equation*,

$$x_{n+1} = kx_n(1 - x_n),$$

where x_n is the population, on a scale of 0 to 1, at a certain time and x_{n+1} is the population one unit of time later. The starting point, x_0, is a number between 0 and 1. The number k represents the reproductivity of the species in question; it is a *parameter* of the model.

This is an iterative process and the outcome depends on the value of k. For $0 \leqslant k \leqslant 1$, the values of x get progressively smaller and converge to zero. The population dies out, whatever the starting value. Table 5(a) illustrates this in the case $k = 0.3$ with starting value $x_0 = 0.5$.

For values of k between 1 and 3, the values of x converge to a particular value which depends on the value of k but not on the starting point. Thus when $k = 2.2$, x converges to 0.545 454 … , as shown in Table 5(b). The population assumes a stable level.

$k = 0.3$	
n	x_n
0	0.5
1	0.075
2	0.020 812 …
3	0.006 113 …
4	0.001 822 …
5	0.000 545 …
6	0.000 163 …
7	0.000 049 …
8	0.000 014 …
9	0.000 004 …
10	0.000 001 …
11	0.000 000 …

Table 5(a)

$k = 2.2$	
n	x_n
0	0.5
1	0.55
2	0.544 5
3	0.545 643 …
4	0.545 416 …
5	0.545 462 …
6	0.545 453 …
7	0.545 454 …
8	0.545 454 …
9	0.545 454 …
10	0.545 454 …
11	0.545 454 …

Table 5(b)

For values of k a little bigger than 3, the value of x ends up oscillating between two outcomes. Thus when $k = 3.2$, the value of x ends up oscillating between 0.513… and 0.799… . The population goes up and down regularly.

For slightly larger values of k, the value of x ends up oscillating between 4 or 8 outcomes.

Fig. 6 shows the outcomes for values of k between 1 and 3.55.

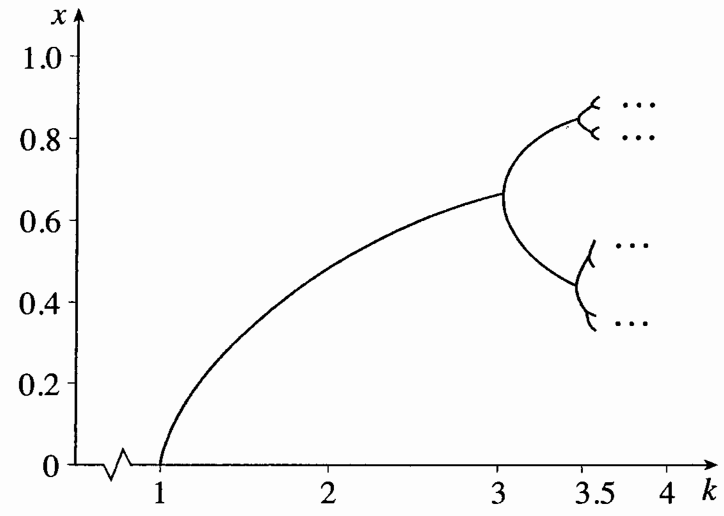

Fig. 6

Larger values of k than those illustrated in Fig. 6 produce even more outcomes, 16, 32 etc. The number of outcomes is always a power of 2.

For still larger values of k there is no pattern at all. The system goes into chaos.

When $k = 3$, there is a change of regime from one non-zero outcome to two. This is called a *point of bifurcation*.

The next point of bifurcation occurs when the number of outcomes changes from 2 to 4; it occurs at $k = 3.4485$ (to 4 decimal places). The one after that is at $k = 3.5437$ when the change is from 4 to 8, and so on.

Number of non-zero outcomes	**1**	**2**	**4**
Values of k	1 to 3	3 to 3.4485	3.4485 to 3.5437
Length of present interval	2	0.4485	0.0952
$\dfrac{\textbf{Previous interval}}{\textbf{Present interval}}$	–	$\dfrac{2}{0.4485} = 4.459\ldots$	$\dfrac{0.4485}{0.0952} = 4.711\ldots$

Table 7

Table 7 shows the intervals, in terms of the parameter k, for the first 3 regimes (excluding $0 \leqslant k \leqslant 1$ for which the population dies out). The final row gives the ratios of lengths of consecutive intervals.

What Feigenbaum discovered is that the sequence formed by the values of this ratio converges to a particular number, 4.669 201… . (Australian mathematicians have now computed it to 1000 significant figures.)

What is remarkable about Feigenbaum's number is that it arises not just in the iterative equation described above, but in a wide variety of equations modelling real-life situations which change from order into chaos. It is a universal constant.

For example, an entirely different iterative equation which generates Feigenbaum's number is

$$x_{n+1} = k\sin(\pi x_n).$$

The fact that 30 years ago we did not know about Feigenbaum's number, but do now, illustrates the point that it is a marker in our technological and cultural development. Its discovery depended on having the power of electronic calculation. The number and length of the calculations involved in its discovery were enormous.

So sending the first few digits of Feigenbaum's number to another civilisation could be taken as asking the question "Have you developed computers yet?" However, a positive response could also mean that those in the other civilisation have wonderfully good brains of their own, so good that they do not need computers.

Scientific questions

It would be a matter of great interest to scientists to find out whether quantities we believe to be constant really are: for example the velocity of light, $2.997\,924 \times 10^5$ kilometres per second. Even a small difference in this would require a fundamental re-think of the laws of physics.

Unfortunately this value uses units that are derived from conditions on Earth. The kilometre is approximately $\frac{1}{40\,000}$ of the circumference of the Earth, and the second is approximately $\frac{1}{60 \times 60 \times 24}$ of the time it takes the Earth to spin once on its axis (i.e. one day). Although both of these units are now defined more precisely using basic properties of matter, they retain essentially the same values and so would be meaningless to another civilisation.

It may be that the best that scientists could achieve would be certain ratios. For example

$$\frac{\text{Rest mass of proton}}{\text{Rest mass of electron}} = 1836.108.$$

The numbers scientists would send would almost certainly have been determined experimentally, to a known level of accuracy. If the reply came back with the same number but given to a much higher level of accuracy, it would be a good indication that the other civilisation is more technologically advanced than we are.

Thus it may be that the answers to science-based questions would tell us more about the civilisation's level of development than about science.

Conclusion

There are many other numbers that could be used in this context. No doubt a selection committee would be needed! The numbers mentioned in this article illustrate some of the principles which might guide the work of such a committee.

Mark Scheme 2603
June 2005

2603

MEI P3 June 2005 Mark Scheme post-coordination

General Instructions

1. (a) Please mark in red and award part marks on the right side of the script, level with the work that has earned them.
 (b) If a part of a question is completely correct, or only *one* accuracy mark has been lost, the total mark or slightly reduced mark should be put in the margin at the end of the section, shown as, for example, 7 or 7 − 1, without any ringing. Otherwise, part marks should be shown as in the mark scheme, as M1, A1, B1, etc.
 (c) The total mark for the question should be put in the right hand margin at the end of each question, and ringed.

2. Every page of the script should show evidence that it has been assessed, even if the work has scored no marks.

3. Do not assume that, because an answer is correct, so is the intermediate working; nor that, because an answer is wrong, no marks have been earned.

4. Errors, slips, etc. should be marked clearly where they first occur by underlining or ringing. Missing work should be indicated by a caret (∧).
   - For correct work, use ✓,
   - For incorrect work, use X,
   - For correct work after and error, use ✓
   - For error in follow through work, use ✗

5. An 'M' mark is earned for a correct method (or equivalent method) for that part of the question. A method may contain incorrect working, but there must be sufficient evidence that, if correct, it would have given the correct answer.

   An 'A' mark is earned for accuracy, but cannot be awarded if the corresponding M mark has not be earned. An A mark shown as A1 f.t. or A1 ✓ shows that the mark has been awarded following through on a previous error.

   A 'B' mark is an accuracy mark awarded independently of any M mark.

   'E' marks are accuracy marks dependent on an M mark, used as a reminder that the answer has been given in the question and must be fully justified.

6. If a question is misread or misunderstood in such a way that the nature and difficulty of the question is unaltered, follow the work through, awarding all marks earned, but deducting one mark once, shown as MR − 1, from any accuracy or independent marks earned in the affected work. If the question is made easier by the misread, then deduct more marks appropriately.

7. Mark deleted work if it has not been replaced. If it has been replaced, ignore the deleted work and mark the replacement.

8. Other abbreviations:

c.a.o.	: correct answer only
b.o.d.	: benefit of doubt (where full work is not shown)
X | X	: work of no mark value between crosses
s.o.i.	: seen or implied
s.c.	: special case (as defined in the mark scheme)
w.w.w	: without wrong working

Procedure

1. Before the Examiners' Meeting, mark at least 10 scripts of different standards and bring them with you to the meeting. List any problems which have occurred or that you can foresee.

2. After the meeting, mark 7 scripts and the 3 photocopied scripts provided and send these to your team leader. Keep a record of the marks, and enclose with your scripts a stamped addressed envelope for their return. Your team leader will contact you by telephone or email as soon as possible with any comments. You must ensure that the corrected marks are entered on to the mark sheet.

3. By a date agreed at the standardisation meeting prior to the batch 1 date, send a further sample of about 40 scripts, from complete centres. You should record the marks for these scripts on your marksheets. They will not be returned to you, but you will receive feedback on them. If all is well, you will then be given clearance to send your batch 1 scripts and marksheets to Cambridge.

4. Towards the end of the marking period, your team leader will request a final sample of about 60 scripts. This sample will consist of complete centres and will not be returned to you. The marks must be entered on the mark sheets before sending the scripts, and should be sent, with the remainder of your marksheets, to the office by the final deadline.

5. Please contact your team leader by telephone or email in case of difficulty. Contact addresses and telephone numbers will be found in your examiner packs.

1 (a) $5\cos x - 6\sin x = R\cos(x+\alpha)$ $= R(\cos x\cos\alpha - \sin x\sin\alpha)$ $\Rightarrow R\cos\alpha = 5,\ R\sin\alpha = 6$ $\Rightarrow R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 6^2 = 61$ $\Rightarrow R^2 = 61,\ R = \sqrt{61} = 7.81$ $\tan\alpha = R\sin\alpha / R\cos\alpha = 6/5$ $\Rightarrow \alpha = 0.876$ Accept 0.88 or 0.279π but not $50.2°$	M1 A1 M1 A1cso [4]	Expansion or equations in R and α, soi NB $R(\cos x\cos\alpha + \sin x\sin\alpha)$ gains M1 but final A1 cso will be lost. $R = 7.81$ or $\sqrt{61}$. Accept 7.8 $\tan\alpha = 6/5$ soi Note, $\cos\alpha = 5$, $\sin\alpha = 6 \Rightarrow \tan\alpha = 6/5$ is M0.
(b) $\int_0^{\frac{1}{4}\pi} x\sin 2x\,\mathrm{d}x$ let $u = x$, $\mathrm{d}v/\mathrm{d}x = \sin 2x$ $\Rightarrow v = -\frac{1}{2}\cos 2x$ $= \left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{1}{4}\pi} - \int_0^{\frac{1}{4}\pi}(-\frac{1}{2}\cos 2x).1.dx$ $= 0 + \left[\frac{1}{4}\sin 2x\right]_0^{\frac{1}{4}\pi}$ $= \frac{1}{4}$	M1 A1 A1ft DM1 A1cao [5]	$u = x$, $\mathrm{d}v/\mathrm{d}x = \sin 2x$ $v = -\frac{1}{2}\cos 2x$ $-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x$ ft their v substituting limits
(c) $1 + \ln 1 = 1$, $\sin \pi/2 = 1$, so $(\pi/2, 1)$ lies on curve. $\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = \cos x$ $\Rightarrow \frac{dy}{dx}(1 + \frac{1}{y}) = \cos x$ $\Rightarrow \frac{dy}{dx} = \frac{\cos x}{1 + 1/y} = \frac{y\cos x}{y+1}$ * Allow verification: Separating the variables M1 $y + \ln y = \sin x + C$ A1 $x = \pi/2, y = 1 \Rightarrow C = 0$ E1 When $x = \pi/2$, $y = 1$, $\mathrm{d}y/\mathrm{d}x = 0$	E1 M1 B1 E1 B1 [5] [14]	$1 + \ln 1 = 1$ or $1 + 0 = 1$ $\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx}$ .Some evidence of implicit differentiation. Ignore spurious dy/dx. Allow $\frac{dy}{dx} + y\frac{dy}{dx} =$,or $1 + \frac{1}{y}\frac{dy}{dx} = ...$ but not $1 + y\frac{dy}{dx} = \ldots$ $\cos x$ An intermediate step must be seen. www.

2 (i) $\frac{5}{(2+x)(1-2x)} = \frac{A}{2+x} + \frac{B}{1-2x}$ $\Rightarrow \quad 5 = A(1-2x) + B(2+x)$ $x = -2 \Rightarrow 5 = 5A \;, \Rightarrow A = 1$ $x = ½ \Rightarrow 5 = 2½\, B \;\Rightarrow B = 2$ $\Rightarrow \quad \frac{5}{(2+x)(1-2x)} = \frac{1}{2+x} + \frac{2}{1-2x}$	M1 A1 A1 [3]	Equating numerators soi $A = 1/(2+x)$ $B = 2/(1-2x)$ SC If no working seen , one fraction correct B2, the other correct B1.
(ii) $\frac{dy}{dx} = \frac{5y}{(2+x)(1-2x)}$ $\Rightarrow \quad \int \frac{1}{y}dy = \int \frac{5}{(2+x)(1-2x)}dx$ $= \int (\frac{1}{2+x} + \frac{2}{1-2x})dx$ $\Rightarrow \quad \ln y = \ln\|2+x\| - \ln\|1-2x\| + c$ When $x = 0$, $\ln 2 = \ln 2 + c \;\Rightarrow c = 0$ $\Rightarrow \quad \ln y = \ln(2+x) - \ln(1-2x) = \ln(\frac{2+x}{1-2x})$ $\Rightarrow \quad y = \frac{2+x}{1-2x}$ *	M1 B1 B1 B1 DM1 E1 [6]	Re-arranging. Allow eg $\int y\, dy = \ldots$ $\ln y$) $\ln(2+x)$) Condone no C $-\ln(1-2x)$) calculating c without incorrect log work eg $y = \frac{2+x}{1-2x} + e^c \Rightarrow c = 0$ is DM0 www.
(iii) $y = \frac{2+x}{1-2x} = (2+x)(1-2x)^{-1}$ $= (2+x)(1+2x+4x^2+8x^3+\ldots)$ $= 2+4x+8x^2+16x^3$ $+ \; x+2x^2+4x^3+\ldots$ $= 2+5x+10x^2+20x^3+\ldots$ * Valid for $\|x\| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$	M1 M1 E1 B1 [4]	Binomial expansion : evidence of correct binomial coefficients and correct use of $(-2x)^r$ multiplying out given answer
(iv) $f'(x) = 5 + 20x + 60x^2$ $f'(0.01) = 5.206$ $y = \frac{2.01}{0.98} = 2.051\ldots$ $\frac{dy}{dx} = \frac{5 \times 2.051\ldots}{0.98 \times 2.01} = 5.206\cdot\cdot$ So accurate to 3 decimal places (OR $\frac{dy}{dx} = \frac{5}{(1-2x)^2} = \frac{5}{(1-0.02)^2}$ using the	B1 M1 A1 [3]	5.206 substituting for x in y,and then x and y in dy/dx 5.206

quotient rule, (must be correct), and substitution, = 5.206)	[16]	

3 (i) At A, $\cos 2\theta = -1$, $\Rightarrow$ A is (2, 0) At B, $\sin 2\theta = 1$ $\Rightarrow$ B is (1, 2).	M1 A1 M1 A1 [4]	Or, $y = 0 \Rightarrow \sin 2\theta = 0$ $\theta = \pi/2, x=2$ or, $x = 1 \Rightarrow \cos 2\theta = 0$ $\theta = \pi/4, y=2$ SC Allow B2, B2 for correct answers without working
(ii) $y = x \Rightarrow 1 - \cos 2\theta = 2 \sin 2\theta$ $\Rightarrow 2 \sin^2\theta = 4 \sin\theta \cos\theta$ $\Rightarrow 2 \sin\theta(\sin\theta - 2\cos\theta) = 0$ $\Rightarrow (\sin\theta = 0)$ or $\sin\theta = 2\cos\theta$ $\Rightarrow \tan\theta = 2$ *	M1 M1 A1 E1 [4]	Equating Attempt to use the double angle formulae to obtain an equation in θ. Any correct equation in θ
(iii) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ $= \frac{4\cos 2\theta}{2\sin 2\theta}$ $= 2\cot 2\theta$* $\Rightarrow$ gradient at C = $2.\frac{1-\tan^2\theta}{2\tan\theta}$ $= 2\times\frac{(-3)}{4}$ $= -1.5$	M1 A1 E1 M1 A1cao [5]	$\frac{dy}{dx} = \frac{their dy/d\theta}{their dx/d\theta}$ $\frac{4\cos 2\theta}{2\sin 2\theta}$ (Allow these marks if seen in part (i).) Use of $\tan 2\theta$ formula or $\theta = 63.43\ldots°$ or 1.107… rads
(iv) $\cos 2\theta = 1 - x$, $\sin 2\theta = y/2$ $\Rightarrow \cos^2 2\theta + \sin^2 2\theta = (1-x)^2 + \frac{y^2}{4} = 1$ or eg $y^2 + 4x^2 - 8x = 0$	M1 A1 [2] [15]	These equations plus a valid method of eliminating 2θ

4 (i) $\mathbf{a} = \begin{pmatrix} -3 \\ 4 \\ 12 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$ $\Rightarrow \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert} = \frac{(-3)\times 2 + 4\times 4 + 12\times 4}{\sqrt{169}\times\sqrt{36}}$ $= \frac{58}{78} \Rightarrow \theta = 41.96°$ (Accept 42° or 0.73 radians) Area of triangle OAB = $\frac{1}{2}\times 13\times 6\sin\theta$ = 26.1 (units ^{2}) (Accept 26)	M1 A1 A1 A1 M1 A1 ft [6]	use of scalar product (allow one slip) correct numerator correct denominator ft their θ
(ii) $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \\ 12 \end{pmatrix} + \lambda\begin{pmatrix} 8 \\ -9 \\ 5 \end{pmatrix}$	B1 B1 [2]	$\mathbf{r} = \begin{pmatrix} -3 \\ 4 \\ 12 \end{pmatrix} + \ldots$ or, using D, $\mathbf{r} = \begin{pmatrix} 5 \\ -5 \\ 17 \end{pmatrix} + \ldots$ $\ldots + \lambda\begin{pmatrix} 8 \\ -9 \\ 5 \end{pmatrix}$ o.e.
(iii) $\mathbf{c}.\overrightarrow{OA} = \begin{pmatrix} 8 \\ -9 \\ 5 \end{pmatrix}.\begin{pmatrix} -3 \\ 4 \\ 12 \end{pmatrix} = -24 - 36 + 60 = 0$ $\mathbf{c}.\overrightarrow{OB} = \begin{pmatrix} 8 \\ -9 \\ 5 \end{pmatrix}.\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} = 16 - 36 + 20 = 0$ $\Rightarrow$ **c** is perpendicular to the plane OAB OAB: $8x - 9y + 5z = 0$ CDE: $8x - 9y + 5z = d$ At C, $x = 8$, $y = -9$, $z = 5$ $\Rightarrow d = 64 + 81 + 25 = 170$ $\Rightarrow 8x - 9y + 5z = 170$	B1 B1 B1 M1 A1 [5]	Working must be seen
(iv) OC = √170 Volume of prism = 26.1 × √170 = 340 (units ^{3})	M1 A1cao [2] [15]	

Section B 1. The masses are measured in units. The ratio is dimensionless	B1 B1 [2]	'units cancel out' $\Rightarrow$ B2
2. Converting from base 5, $3.03232 = 3 + \frac{0}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \frac{2}{5^5}$ $= 3.14144$	M1 A1 [2]	Allow this M1 for misreading and $3.03232 = 3 + \frac{0}{6} + \frac{3}{6^2} + \frac{2}{6^3} + \frac{3}{6^4} + \frac{2}{6^5}$ then A0
3. n $\quad x_n$ 0 $\quad$ 0.5 1 $\quad$ 0.8 2 $\quad$ 0.512 3 $\quad$ **0.7995392** 4 $\quad$ **0.5128840565** 5 $\quad$ **0.7994688035**	B1 [1]	Ignore the ninth and tenth d.p.
4. $\frac{\phi}{1} = \frac{1}{\phi - 1}$ $\Rightarrow \phi^2 - \phi = 1 \Rightarrow \phi^2 - \phi - 1 = 0$ Using the quadratic formula gives $\phi = \frac{1 \pm \sqrt{5}}{2}$	M1 E1 [2]	A Q.E. in ϕ, (either form), and an attempt to solve- formula must be used correctly. SC Allow B2 for verification if completely correct.
5. Let $r = \frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}}$ $a_{n+1} = 2a_n + 3a_{n-1}$ dividing through by a_n. $\Rightarrow r = 2 + \frac{3}{r}$ $\Rightarrow r^2 - 2r - 3 = 0$ $\Rightarrow (r-3)(r+1) = 0$ $\Rightarrow$ $r = 3$ (discounting -1)	M1 M1 A1 A1 [4]	Either ratio Forming the equation in r, $r = 2 + \frac{3}{r}$ Correct equation in standard QE form Correct solution by factorisation or formula and rejecting the –ve root SC Allow B2 for calculation at least as far as $a_8 = 1093$, $a_9 = 3281$, $a_{10} = 9841$ $\Rightarrow a_9/a_8 = 3.00\ldots$, $a_{10}/a_9 = 2.99\ldots$

6. The length of the next interval = l, where $$\frac{0.0952...}{l} = 4.669...$$ $\Rightarrow \quad l = 0.0203…$ So next bifurcation at 3.5437… + 0.0203… ≈ 3.564	M1 A1 M1 A1 [4] [15]	

2603 - Pure Mathematics 3

General Comments

There was a wide range of marks for this paper, fewer candidates than in past years scoring in the range 65 plus and more than previously receiving marks of 15 or less.
The majority of candidates scored marks between these two extremes. Marks were pulled down by poorer performances on the comprehension paper on which some 50 per cent of candidates scored 6 marks or fewer. Having said that, there were some very good performances and quite a pleasing number of candidates scored well on both papers. These candidates presented their work well, in contrast to weaker candidates whose work was often very difficult to follow and, occasionally, difficult even to read.

Marks were often lost by candidates who missed, or misread, small parts of questions. For example an appreciable number of candidates failed to find the area of the triangle in question 4(i), and quite a large number found f(0.01) instead of f′(0.01) in question 2(iv). There was little evidence of candidates being short of time unless time had been wasted with overlong solutions or work crossed out and repeated with little improvement.

Comments on Individual Questions

1) (a) This question was generally well answered although the value of α was frequently given in degrees. Other errors that occurred were $\tan\alpha = 5/6$ or $-6/5$ or $\alpha = 0.88\pi$.

(b) Almost all candidates realized that integration by parts was required and chose the correct functions, $u = x$ and $dv/dx = \sin 2x$. Integration of the latter function however, was not so sure; $v = ½ \cos 2x$ or $v = - 2 \cos 2x$, or similar, were often seen. These errors were often repeated or introduced at the next stage when integrating v. Most candidates arriving at a result of integration were able to substitute the limits correctly, although $1/2x = 45°$ at the upper limit was seen occasionally.

(c) Most candidates were able to show that the point $(\pi/2, 1)$ lies on the given curve.
Solutions to the main part of the question involving implicit differentiation were often spoiled by the omission of brackets,

$$1 + \frac{1}{y}\frac{dy}{dx} = \cos x,$$

by attempts to fudge the result after an error,

$$y + \frac{1}{y}\frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{y + \frac{1}{y}} = \frac{\cos x}{\frac{y+1}{y}} = \frac{y\cos x}{y+1},$$

or by poor notation; the misuse of the symbol $\frac{dy}{dx}$ was often seen.

In the final part of this question it was surprising to find candidates interpreting $\frac{0}{2}$ as undefined or infinity.

2) **Partial fractions, differential equations, the binomial theorem**

(i) The partial fractions were obtained correctly by almost all candidates, just a few errors in the coefficients. For example, 5A=5 ⇒A=5 was seen on more than one occasion.

(ii) The differential equation was not so well done, the separation of variables often being incorrect or completely absent. Candidates realized that the partial fractions had to be used and so the most common errors were,

$$\int 5y\,dy = \int (\frac{1}{2+x} + \frac{2}{1-2x})dx\,;$$

or, simply, $$y = \int (\frac{1}{2+x} + \frac{2}{1-2x})dx\,;$$

although dy and dx were often omitted. This meant that most candidates had the opportunity to score two marks at least for integrating their partial fractions. For those candidates who separated the variables correctly and integrated, the next common errors were the omission of an arbitrary constant or incorrect log work in the process of finding the value of the constant,

$$\ln y = \ln(2+x) - \ln(1-2x) + C$$

$$\Rightarrow \quad y = \frac{2+x}{1-2x} + C \text{, or perhaps } +e^C.$$

(iii) Many candidates were able to apply the binomial theorem to obtain a correct expansion of $(1-2x)^{-1}$.Surprisingly, over-enthusiasm for the binomial theorem led a number of candidates to apply it, also, to 2+x

$$\begin{aligned} 2+x &= 2(1+x/2)^1 \\ &= 2(1 + 1.x/2 + \frac{1.(1-1)}{2}(x/2)^2 + \ldots.) \\ &= 2 + 2.x/2 = 2+x \end{aligned}$$

Multiplication of the series by 2+x was almost always correct, although just occasionally the x^3 term was omitted from the binomial expansion leading to an incorrect x^3 term in the product.

Unfortunately a very large number of candidates lost the final mark in this part by failing to state the range of validity or by giving it incorrectly; $|x| \le ½$, $x < ½$,or $½ < x < -1/2$, etc.

(iv) Many candidates differentiated the given f(x) and substituted x = 0.01 to find f′(0.01) correctly but then failed to find the value of dy/dx at x=0.01, for comparison . Others found f(0.01) and y(0.01) both equal to 2.051 and thought they had answered the question. There were, however, many fully correct answers.

3) **Parametric coordinates and equations**

(i) A variety of methods was seen often leading to the correct coordinates for A and B. The work of many candidates, however, was confused, and this question was a good example of one where the work of weaker candidates was very difficult to follow because of the lack of explanation. In some cases a value of θ was given as one of the coordinates.

(ii) Almost all candidates wrote down the equation $1 - \cos 2\theta = 2 \sin 2\theta$ as a start to this question and realized that it was then necessary to use the double angle formulae in order to prove the given result. The RHS of the above equation presented no problem, and many good candidates gave a very neat solution using $\frac{1}{2}(1 - \cos 2\theta) = 2 \sin^2 \theta$, but weaker candidates often used an incorrect formula for $\cos 2\theta$ or substituted with little regard for bracket or signs.

$\cos 2\theta = \cos^2\theta - 1$ or $1 - \sin^2\theta$,

$\cos 2\theta = 1 - \cos^2\theta - \sin^2\theta$,

$\cos 2\theta = 1 - (\cos^2\theta - \sin^2\theta) = 1 - \cos^2\theta + \sin^2\theta = 1 - 1 = 0$

$\cos 2\theta = 1 - (1 - 2 \sin^2\theta) = -2 \sin^2\theta$

were all seen.

(iii) Some candidates did the work for the start of this question in part (i) and sensibly referred back to it, others repeated the work here. Many candidates obtained full marks for dy/dx. Many then went on to find the gradient at C, finding $\tan^{-1}2$ on their calculator and retaining it to obtain an exact value of -1.5 . Others used the double angle formula for $\tan 2\theta$, usually correctly.

(iv) Only a small number of candidates obtained the Cartesian equation using the identity $\sin^2 2\theta + \cos^2 2\theta = 1$; most attempted to use $\sin^2 \theta + \cos^2\theta = 1$ which involved more work and therefore more likelihood of errors occurring.

4) **Vectors**

(i) The first part of (i) was answered well by the majority of candidates but many failed to go on to find the area of triangle OAB. Those who did attempt this often used an incorrect method; ½ |AB| sin θ, ½ $\underline{a}.\underline{b}$ cos θ, ½

OB.BA and ½ OA. AB were all seen, as well as incorrect attempts to find a perpendicular height to go with base OB or OA.

(ii) Many candidates confused the vector equation of AD with the vector AD and this may account for

$$\text{AD} = \begin{pmatrix} -3 \\ 4 \\ 12 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -9 \\ 5 \end{pmatrix}$$

instead of $\underline{r}$ = ……

(iii) The fact that $\underline{c}.\underline{a} = 0$ implies that $\underline{c}$ is perpendicular to $\underline{a}$ was well known and usually shown; just a few candidates failed to show the working which was necessary to obtain the mark. However many candidates thought that this was sufficient and did not show that in addition $\underline{c}.\underline{b} = 0$ or $\underline{c}.\underline{AB} = 0$. The equation of the plane OAB was usually found correctly but not always the equation of CDE. A number of candidates wasted time on this question by attempting to use the vector equation of a plane and eliminate the parameters to find the Cartesian equation. Such attempts were only very rarely successful.

(iv) Most candidates obtained the method mark for this final question but many, not having the correct area of the triangle OAB could not get the correct answer.

Section B Comprehension

1) A reasonable number of candidates recognized the problem with units and explained that sending the ratio of the two masses was a way of overcoming this. Other candidates commented on ‘the different conditions on earth’ or the ratio being ‘the same everywhere’ without saying why. Some candidates erroneously referred to different number systems or numbers being too small to transmit.

2) Many candidates were able to write down the correct expression for the conversion of the given number in base 5 to a decimal number but unfortunately many of them failed to evaluate it. Many other candidates just wrote down the usual value of π in terms of powers of 10.

3) Very well done indeed, very few candidates were unable to complete the table correctly.

4) Candidates were roughly, equally divided between those who verified that the given values of ϕ satisfied the equation, and those who transformed the equation into standard form and then used the formula to solve it. Unfortunately many of the former used their calculator to evaluate the given values of ϕ and then substituted them into the equation. This approximate solution was not acceptable

but a complete justification involving rationalizing was accepted. Those candidates transforming the equation usually applied the formula correctly.

5) Many candidates took the reference to ‘the method on page 5’ to refer to the demonstration that the sequence there appeared to converge to the limit ϕ, rather than to the proof that it did converge. Some credit was allowed to those candidates who calculated a sufficient number of ratios to draw a reasonable conclusion, but very few did this, and many made errors in their calculations of the early terms of the sequence. Of those who used the algebraic approach some used inconsistent values of the ratio r, but many obtained the appropriate quadratic equation, solved it correctly, gave the positive root as their answer and rejected the negative root.

6) Many candidates failed to attempt this question but others, not necessarily the most able candidates, completed it correctly. There was sometimes some misunderstanding as to which point of bifurcation was required and some candidates calculated both the point of bifurcation from 8 to 16 outcomes and also the point where the number of outcomes changed from 16 to 32. Some candidates gave a range of values of k rather than a specific value.